



TITLE:

STRONG CONVERGENCE OF ISHIKAWA ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS (Nonlinear Analysis and Convex Analysis)

AUTHOR(S):

Kim, Gang Eun

CITATION:

Kim, Gang Eun. STRONG CONVERGENCE OF ISHIKAWA ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS (Nonlinear Analysis and Convex Analysis). 数理解析研究所講究録 2004, 1365: 176-184

ISSUE DATE:

2004-04

URL:

<http://hdl.handle.net/2433/25350>

RIGHT:

STRONG CONVERGENCE OF ISHIKAWA ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

GANG EUN KIM

Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Ohokayama,
Meguroku, Tokyo 152-8552, Japan

Abstract—Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space. We prove that if $T : C \rightarrow C$ is both compact iterates and asymptotically nonexpansive, the Ishikawa iteration process with errors defined by $x_1 \in C$, $x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n$, and $y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n$ converges strongly to some fixed point of T . This generalizes the recent theorems due to Rhoades [5], Schu [6] and Schu [7].

Keywords—strong convergence, fixed point, Mann and Ishikawa iteration process, asymptotically nonexpansive mapping.

1. Introduction

Let C be a nonempty bounded closed convex subset of a Banach space E and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [1] if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n \rightarrow \infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. In particular, if $k_n = 1$ for all $n \geq 1$, T is said to be *nonexpansive*. T is said to be *uniformly L -Lipschitzian* if there exists a constant $L > 0$, such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. T is said to be compact if it maps bounded sets into relatively compact ones. We denote by $F(T)$ the set of all fixed points of T , i.e., $F(T) = \{x \in C : Tx = x\}$. We also denote by \mathbb{N} the set of all positive integers. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ will denote strong convergence of the sequence $\{x_n\}$ to x . For a mappings T of C into itself, Rhoades [5] considered the following modified Ishikawa iteration process (cf. Ishikawa [3]) in C defined by

$$\begin{aligned} (1) \quad & x_1 \in C, \\ & x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ & y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$. If $\beta_n = 0$ for all $n \geq 1$, then the iteration process (1) becomes the following modified Mann iteration process (cf. Mann [4], Schu [6]):

$$(2) \quad \begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n x_n, \end{aligned}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

Recently, Schu [7] proved that if E is a uniformly convex Banach space, C is a nonempty bounded closed and convex subset of E , and $T : C \rightarrow C$ is an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and T^m is compact for some $m \in \mathbb{N}$, then for any $x_1 \in C$, the sequence $\{x_n\}$ defined by (2), where $\{\alpha_n\}$ is chosen so that $0 < a \leq \alpha_n \leq b < 1$, for all $n \geq 1$ and some $a, b \in (0, 1)$, converges strongly to some fixed point of T . This extended a result of Schu [6] to uniformly convex Banach spaces. On the other hand, Rhoades [5] proved that if E is a uniformly convex Banach space, C is a nonempty bounded closed convex subset of E , and $T : C \rightarrow C$ is a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n^r - 1) < \infty$, $r = \max\{2, p\}$, then for any $x_1 \in C$, the sequence $\{x_n\}$ defined by (1), where $\{\alpha_n\}, \{\beta_n\}$ satisfy $a \leq (1 - \alpha_n), (1 - \beta_n) \leq 1 - a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T . We consider a more general iterative process of the type (cf. Xu [10]) emphasizing the randomness of errors as follows:

$$(3) \quad \begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n &= \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ are two sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 1$,
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

If $\gamma_n = \gamma'_n = 0$ for all $n \geq 1$, then the iteration process (3) reduces to the Ishikawa iteration process [3], while setting $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \geq 1$, (3) reduces to the Mann iteration process with errors, which is a generalized case of the Mann iteration process [4].

In this paper, we prove strong convergence theorems of the Ishikawa (and Mann) iteration process with errors defined by (3) for a compact iterates and asymptotically nonexpansive mapping in a uniformly convex Banach space, which generalize the recent theorems due to Rhoades [5], Schu [6] and Schu [7].

2. Strong convergence theorems

We first begin with the following:

Lemma 1 [9]. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. Then $\lim_{n \rightarrow \infty} a_n$ exists.

Lemma 2 [2]. Let E be a uniformly convex Banach space. Let $x, y \in E$. If $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon > 0$, then $\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$ for λ with $0 \leq \lambda \leq 1$.

Lemma 3 (cf. [6]). Let E be a normed space and let C be a nonempty bounded convex subset of E . Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian mapping. Define the sequence $\{x_n\}$ defined by (3). Set $w_n = \|T^n x_n - x_n\|$, for all $n \geq 1$. Then

$$\|x_n - Tx_n\| \leq w_n + L(2 + 2L + L^2)w_{n-1} + L^2(1 + L)M^*\gamma'_{n-1} + L(1 + L)M^*\gamma_{n-1},$$

for all $n \geq 1$, where $M^* := \sup_{n \geq 1} \|x_n - u_n\| \vee \sup_{n \geq 1} \|x_n - v_n\| < \infty$.

Proof. Since

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - x_n\| \\ &\leq \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|v_n - x_n\| \\ &\leq w_n + \gamma'_n M^*, \end{aligned}$$

$$\begin{aligned} \|T^n y_n - x_n\| &\leq \|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\leq L\|y_n - x_n\| + w_n \\ &\leq L\{w_n + \gamma'_n M^*\} + w_n \\ &= (1 + L)w_n + LM^*\gamma'_n \end{aligned}$$

and thus

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|\alpha_{n-1}x_{n-1} + \beta_{n-1}T^{n-1}y_{n-1} + \gamma_{n-1}u_{n-1} - x_{n-1}\| \\ &\leq \beta_{n-1}\|T^{n-1}y_{n-1} - x_{n-1}\| + \gamma_{n-1}\|u_{n-1} - x_{n-1}\| \\ &\leq (1 + L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}, \end{aligned}$$

$$\begin{aligned} \|T^{n-1}x_n - x_n\| &\leq \|T^{n-1}x_n - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &\leq w_{n-1} + (1 + L)\|x_n - x_{n-1}\| \\ &\leq w_{n-1} + (1 + L)\{(1 + L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}\}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ &\leq w_n + L\|T^{n-1}x_n - x_n\| \\ &\leq w_n + L[w_{n-1} + (1 + L)\{(1 + L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}\}] \\ &= w_n + L(2 + 2L + L^2)w_{n-1} + L^2(1 + L)M^*\gamma'_{n-1} + L(1 + L)M^*\gamma_{n-1}. \end{aligned}$$

□

Using Lemma 1, we have the following:

Lemma 4. Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that the sequence $\{x_n\}$ defined by (3). Then $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists, for any $z \in F(T)$.

Proof. The existence of a fixed point of T follows from Goebel-Kirk [1]. For a fixed $z \in F(T)$, since $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := \sup_{n \geq 1} \|x_n - z\| \vee \sup_{n \geq 1} \|u_n - z\| \vee \sup_{n \geq 1} \|v_n - z\| < \infty.$$

Put $c_n = k_n - 1$. Since

$$\begin{aligned} \|T^n y_n - z\| &\leq k_n \|y_n - z\| \\ &= (1 + c_n) \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - z\| \\ &\leq (1 + c_n) \{\alpha'_n \|x_n - z\| + \beta'_n \|T^n x_n - z\| + \gamma'_n \|v_n - z\|\} \\ &\leq (1 + c_n) \{\alpha'_n \|x_n - z\| + \beta'_n (1 + c_n) \|x_n - z\| + \gamma'_n \|v_n - z\|\} \\ &\leq (1 + c_n) \{\alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + c_n \|x_n - z\| + \gamma'_n \|v_n - z\|\} \\ &= \alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + c_n \|x_n - z\| + \gamma'_n \|v_n - z\| \\ &\quad + c_n \{\alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + c_n \|x_n - z\| + \gamma'_n \|v_n - z\|\} \\ &\leq (1 - \gamma'_n) \|x_n - z\| + 4M c_n + M \gamma'_n, \end{aligned}$$

we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|T^n y_n - z\| + \gamma_n \|u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \{(1 - \gamma'_n) \|x_n - z\| + 4M c_n + M \gamma'_n\} + \gamma_n M \\ &= (1 - (\gamma_n + \beta_n \gamma'_n)) \|x_n - z\| + 4M \beta_n c_n + M(\gamma_n + \beta_n \gamma'_n) \\ &\leq \|x_n - z\| + 4M c_n + M(\gamma_n + \gamma'_n). \end{aligned}$$

By Lemma 1, we readily see that $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. \square

By using Lemma 1–Lemma 4, we have the following:

Theorem 1. Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $\{k_n\}$

satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose $x_1 \in C$, and the sequence $\{x_n\}$ defined by

(3) satisfies $0 < a \leq \alpha_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \leq \beta'_n \leq b < 1$ for all $n \geq 1$ and some

$a, b \in (0, 1)$ or $0 < a \leq \beta_n \leq 1$, $0 < a \leq \alpha'_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Then $\liminf_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Proof. The existence of a fixed point of T follows from Goebel-Kirk [1]. For a fixed $z \in F(T)$, since $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := \sup_{n \geq 1} \|x_n - z\| \vee \sup_{n \geq 1} \|u_n - z\| \vee \sup_{n \geq 1} \|v_n - z\| < \infty.$$

By Lemma 4, we see that $\lim_{n \rightarrow \infty} \|x_n - z\| (\equiv r)$ exists. If $r = 0$, then the conclusion is obvious. So, we assume $r > 0$. Note that $d_n := \max\{\gamma'_n, \gamma_n/a, \gamma'_n/a\} \rightarrow 0$ as $n \rightarrow \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$. Put $c_n = k_n - 1$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, we have

$$(4) \quad \lim_{n \rightarrow \infty} c_n = 0.$$

Since $\|T^n y_n - z\| \leq \|x_n - z\| + 4Mc_n + Md_n$ and

$$\left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq \|x_n - z\| + 4Mc_n + Md_n,$$

by using Lemma 2 and Takahashi [8], we obtain

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\ &= \left\| \beta_n (T^n y_n - z) + (1 - \beta_n) \left(\frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right) \right\| \\ &\leq (\|x_n - z\| + 4Mc_n + Md_n) [1 - 2\beta_n(1 - \beta_n)] \\ &\quad \times \delta_E \left(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n} \right). \end{aligned}$$

Thus, by using $0 < a \leq \alpha_n \leq b < 1$, we obtain

$$\begin{aligned} &2\beta_n a (\|x_n - z\| + 4Mc_n + Md_n) \delta_E \left(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n} \right) \\ &\leq 2\beta_n (1 - \beta_n) (\|x_n - z\| + 4Mc_n + Md_n) \delta_E \left(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n} \right) \\ &\leq \|x_n - z\| - \|x_{n+1} - z\| + 4Mc_n + Md_n. \end{aligned}$$

Since

$$2a \sum_{n=1}^{\infty} \beta_n (\|x_n - z\| + 4Mc_n + Md_n) \delta_E \left(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n} \right) < \infty,$$

$\sup_{n \geq 1} \|T^n y_n - u_n\| < \infty$, and δ_E is strictly increasing and continuous, we obtain

$$(5) \quad \liminf_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0.$$

Since

$$\begin{aligned}
\|T^m x_n - x_n\| &\leq \|T^m x_n - T^m y_n\| + \|T^m y_n - x_n\| \\
&\leq (1 + c_n)\|x_n - y_n\| + \|T^m y_n - x_n\| \\
&= (1 + c_n)\|x_n - \alpha'_n x_n - \beta'_n T^m x_n - \gamma'_n v_n\| + \|T^m y_n - x_n\| \\
&\leq (1 + c_n)\beta'_n\|T^m x_n - x_n\| + (1 + c_n)\gamma'_n\|x_n - v_n\| + \|T^m y_n - x_n\| \\
&\leq (1 + c_n)b\|T^m x_n - x_n\| + (1 + c_n)\gamma'_n\|x_n - v_n\| + \|T^m y_n - x_n\| \\
&= b\|T^m x_n - x_n\| + c_n b\|T^m x_n - x_n\| + (1 + c_n)\gamma'_n\|x_n - v_n\| + \|T^m y_n - x_n\| \\
&\leq b\|T^m x_n - x_n\| + c_n(2 + c_n)b\|x_n - z\| + (1 + c_n)\gamma'_n\|x_n - v_n\| + \|T^m y_n - x_n\|,
\end{aligned}$$

we obtain

$$\begin{aligned}
(1 - b)\|T^m x_n - x_n\| &\leq c_n(2 + c_n)b\|x_n - z\| + (1 + c_n)\gamma'_n\|x_n - v_n\| + \|T^m y_n - x_n\| \\
&\leq c_n(2 + c_n)bM + 2(1 + c_n)\gamma'_n M + \|T^m y_n - x_n\|.
\end{aligned}$$

By using (4) and (5), we obtain

$$(6) \quad \liminf_{n \rightarrow \infty} \|T^m x_n - x_n\| = 0.$$

On the other hand, if $0 < a \leq \beta_n \leq 1$, $0 < a \leq \alpha'_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ for all $n \geq 1$ and some $a, b \in (0, 1)$, then we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^m y_n + \gamma_n u_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n \|T^m y_n - z\| + \gamma_n \|u_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n(1 + c_n)\|y_n - z\| + \gamma_n \|u_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \|y_n - z\| + M\gamma_n \\
&= (1 - \beta_n - \gamma_n)\|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \|y_n - z\| + M\gamma_n \\
&\leq (1 - \beta_n)\|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \|y_n - z\| + M\gamma_n
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\|x_{n+1} - z\| - \|x_n - z\|}{\beta_n} &\leq \|y_n - z\| - \|x_n - z\| + c_n \|y_n - z\| + M \frac{\gamma_n}{a} \\
&\leq \|y_n - z\| - \|x_n - z\| + c_n \{\|x_n - z\| + M c_n + M \gamma'_n\} + M d_n.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_n - z\| - \|y_n - z\| &\leq \frac{\|x_n - z\| - \|x_{n+1} - z\|}{\beta_n} + c_n \{\|x_n - z\| + M c_n + M \gamma'_n\} + M d_n \\
(7) \quad &\leq \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n \{M(1 + c_n) + M \gamma'_n\} + M d_n.
\end{aligned}$$

Since

$$\begin{aligned}\|T^m x_n - z\| &\leq (1 + c_n)\|x_n - z\| \\ &\leq \|x_n - z\| + Mc_n + Md_n\end{aligned}$$

and

$$\left\| \frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right\| \leq \|x_n - z\| + Mc_n + Md_n,$$

we obtain

$$\begin{aligned}\|y_n - z\| &= \|\alpha'_n x_n + \beta'_n T^m x_n + \gamma'_n v_n - z\| \\ &= \left\| \beta'_n (T^m x_n - z) + (1 - \beta'_n) \left(\frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right) \right\| \\ (8) \quad &\leq (\|x_n - z\| + Mc_n + Md_n) \left[1 - 2\beta'_n(1 - \beta'_n) \right. \\ &\quad \left. \times \delta_E \left(\frac{1}{\alpha'_n + \gamma'_n} \cdot \frac{\|\alpha'_n(T^m x_n - x_n) + \gamma'_n(T^m x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n} \right) \right].\end{aligned}$$

By using (7), (8) and $0 < a \leq \alpha'_n \leq b < 1$, we obtain

$$\begin{aligned}&2\beta'_n a (\|x_n - z\| + Mc_n + Md_n) \delta_E \left(\frac{1}{\alpha'_n + \gamma'_n} \cdot \frac{\|\alpha'_n(T^m x_n - x_n) + \gamma'_n(T^m x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n} \right) \\ &\leq 2\beta'_n (1 - \beta'_n) (\|x_n - z\| + Mc_n + Md_n) \delta_E \left(\frac{1}{\alpha'_n + \gamma'_n} \cdot \frac{\|\alpha'_n(T^m x_n - x_n) + \gamma'_n(T^m x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n} \right) \\ &\leq \|x_n - z\| - \|y_n - z\| + Mc_n + Md_n \\ &\leq \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n \{M(1 + c_n) + M\gamma'_n\} + Md_n + Mc_n + Md_n \\ &= \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n \{M(2 + c_n) + M\gamma'_n\} + 2Md_n.\end{aligned}$$

Hence

$$2a \sum_{n=1}^{\infty} \beta'_n (\|x_n - z\| + Mc_n + Md_n) \delta_E \left(\frac{1}{\alpha'_n + \gamma'_n} \cdot \frac{\|\alpha'_n(T^m x_n - x_n) + \gamma'_n(T^m x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n} \right) < \infty.$$

We also obtain (6) similarly to the argument above. By using Lemma 3, we obtain $\liminf_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. \square

Our Theorem 2 improves Theorem 1.5 of Schu [6], Theorem 2.2 of Schu [7] and Theorem 3 of Rhoades [5] to a more general Ishikawa type scheme under much less restrictions on the iterative parameters $\{\alpha_n\}$ and $\{\beta_n\}$.

Theorem 2. Let E be a uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of E , and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and let T^m be compact for some $m \in \mathbb{N}$. If

$x_1 \in C$, and the sequence $\{x_n\}$ defined by (3) satisfies $0 < a \leq \alpha_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \leq \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$ or $0 < a \leq \beta_n \leq 1$, $0 < a \leq \alpha'_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ for all $n \geq 1$ and some $a, b \in (0, 1)$, then $\{x_n\}$ converges strongly to some fixed point of T .

Proof. From Theorem 1, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$(9) \quad \lim_{k \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

Since

$$\begin{aligned} \|T^m x_{n_k} - x_{n_k}\| &\leq \|T^m x_{n_k} - T^{m-1} x_{n_k}\| + \|T^{m-1} x_{n_k} - T^{m-2} x_{n_k}\| + \cdots + \|Tx_{n_k} - x_{n_k}\| \\ &\leq \|Tx_{n_k} - x_{n_k}\| \sum_{j=1}^{m-1} k_j + \|Tx_{n_k} - x_{n_k}\|, \end{aligned}$$

we obtain

$$\lim_{k \rightarrow \infty} \|x_{n_k} - T^m x_{n_k}\| = 0.$$

Since T^m is compact, there exist a subsequence $\{x_{n_{k_i}}\}$ of the sequence $\{x_{n_k}\}$ and a point $p \in C$ such that $x_{n_{k_i}} \rightarrow p$. Thus we obtain $p \in F(T)$ by the continuity of T and (9). Hence we obtain $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$ by Lemma 4. \square

Our Theorem 3 improves Theorem 1.5 of Schu [6], Theorem 2.2 of Schu [7] and Theorem 3 of Rhoades [5] under much less restrictions on the iterative parameters $\{\alpha_n\}$ and $\{\beta_n\}$.

Theorem 3. Let E be a uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of E , and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and let T^m be compact for some $m \in \mathbb{N}$. If

$x_1 \in C$, and the sequence $\{x_n\}$ defined by (1) satisfies $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, $0 \leq \beta_n \leq b < 1$

for all $n \geq 1$ and some $b \in (0, 1)$ or $0 < a \leq \alpha_n \leq 1$, $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$ for all $n \geq 1$ and some $a \in (0, 1)$, then $\{x_n\}$ converges strongly to some fixed point of T .

As a direct consequence, taking $\beta'_n = 0$ and $\gamma'_n = 0$ for $n \in \mathbb{N}$ in Theorem 2, we obtain the following result, which improves Theorem 2.2 of Schu [7] and Theorem 2 of Rhoades [5] under much less restrictions on the iterative parameter $\{\alpha_n\}$.

Theorem 4. Let E be a uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of E , and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and let T^m be compact for some $m \in \mathbb{N}$.

Suppose that $x_1 \in C$, and the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + \beta_n T^m x_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in (0, 1)$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is a sequence in C . Then $\{x_n\}$ converges strongly to some fixed point of T .

Remark. If $\{\alpha_n\}$ is bounded away from both 0 and 1, i.e., $a \leq \alpha_n \leq b$ for all $n \geq 1$ and some $a, b \in (0, 1)$, then $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ hold. However, the converse is not true.

REFERENCES

1. K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171-174.
2. C. W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. Appl. **40** (1972), 369-372.
3. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.
4. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.
5. B. E. Rhoades, *Fixed point iterations for certain nonlinear mappings*, J. Math. Anal. Appl. **183** (1994), 118-120.
6. J. Schu, *Iterative contraction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407-413.
7. J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153-159.
8. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, Yokohama, 2000.
9. K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa Iteration process*, J. Math. Anal. Appl. **178** (1993), 301-308.
10. Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 91-101.